Breakup of a liquid jet

Constantin Weber

TH Dresden

1931

Abstract

The breakup phenomena (drop and wave formation) which can be observed at moderate velocities were investigated theoretically. Rotation-symmetric disturbances lead to drop formation. The breakup time is constant for every liquid and jet thickness at low velocities; the breakup length increases with the velocity. Wave formation is explained by the influence of aerodynamic forces and theoretical laws are derived which agree in character with the experiments.[1]

This is an edited reprint of C. Weber. *On the Breakdown of a Fluid Jet*. Translation. Boulder, CO: University of Colorado, Aug. 1948. 44 pp. OCLC: 18571823. DTIC ATI: 34354. Trans. of "Zum Zerfall eines Flüssigkeitsstrahles [On the disintegration of a liquid jet]". *Zeitschrift für Angewandte Mathematik und Mechanik* 11(2) (Jan. 1, 1931), pp. 136–154. ISSN: 0044-2267. DOI: 10.1002/zamm.19310110207.

The 1948 translation is hard to obtain and partly illegible due to fading. The University of Texas' Interlibrary Services are thanked for producing a maximally clear scan of this translation which made certain parts legible. Some phrases from an alternative translation were used when the 1948 translation was unclear: C. Weber. *Disintegration of a Fluid Jet*. Translation AEC-tr-3783. Oak Ridge National Laboratory, United States Atomic Energy Commission, Nov. 6, 1959. 58 pp. OCLC: 2305825.. Small improvements and corrections were also made as appropriate.

The margin indicates the issue number, page, and column of the original text. Edited by Ben Trettel (https://trettelresearch.com/contact.html). Last updated on 2019-10-20. SVN revision #2369.

[1]Editor's note: There is no abstract in the original. Abstract from J. R. Dulaney. "[Abstract 22672: "Disintegration of a Fluid Jet"]". *Nuclear science abstracts* 13 (24A Dec. 31, 1959), p. 3033. ISSN: 0029-5612. URL: https://hdl.handle.net/2027/mdp.39015026177074?urlappend=%3Bseq=1085.

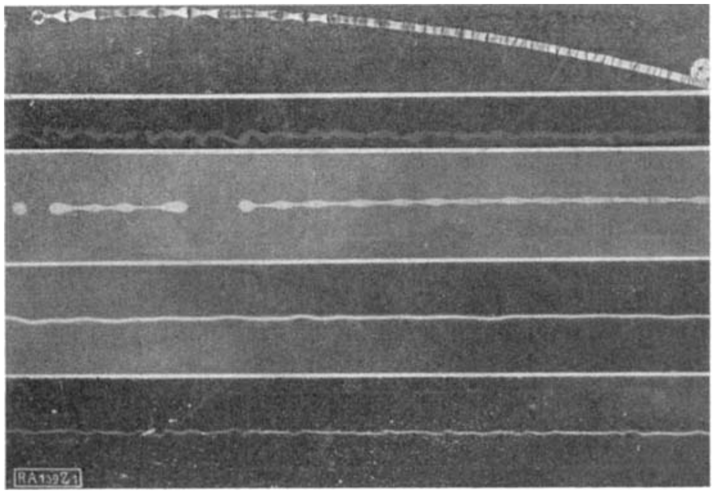

Figure 1: Jet photographs (reduced a and b 1:3, c, d, and e 2:3).

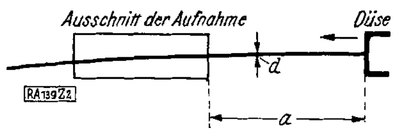

Figure 2: Position of the section of the photograph.
Translator's note: *Ausschnitt der Aufnahme = Section of the photograph. Düse = Nozzle.*

Experiments by A. Haenlein[2] showed various breakup phenomena for a jet of viscous or inviscid fluid released from a nozzle of uniform dimensions. The breakup form depends on the jet velocity and on the physical properties of the liquids, surface tension, density and viscosity. Figure 1 shows photographs of two breakup forms — drops and waves — of water (low viscosity) and of glycerin and castor oil (viscous liquids). Figure 2 represents schematically the position of the section shown by the photographs. p. 136

The experiments prompted me to make a theoretical investigation of the breakup of a jet of an inviscid as well as of a viscous liquid. In 1 the forces caused by surface tension are derived, 2 sets up the derivation of the exact equations, 3 gives the intuitive derivation of an approximate

[2]The Ph.D. study of A. Haenlein, "On the Disintegration of Liquids Jets" will be published in the periodical "Forschung auf dem Gebiet des Ingenieurwesens" 1931, No. 4. [Editor's note: This was translated into English as A. Haenlein. *Disintegration of a Liquid Jet*. Technical memorandum 659. Washington, DC: National Advisory Committee for Aeronautics, Feb. 1932, p. 27. URL: http://ntrs.nasa.gov/search.jsp?R=19930094757. ISSN: 0096-7602.] The study presents numerous further photographs of drop formation, wave formation, transition forms and other breakup forms for very high velocities and sets up theoretical laws for these. The experiments have been performed in the mechanical laboratory of S.T.H. Dresden. Further series of experiments are currently in progress.

equation for drop formation by rotation-symmetric disturbances; in 4 the drop formation of a jet of an inviscid liquid is studied in detail and in 5 that of a jet of a viscous liquid is investigated. Due to the motion of the jet through quiescent air, aerodynamic forces are exerted on the jet surface which are calculated in 6 for the jet with rotation-symmetrical disturbances and with a wavy centerline, respectively. In 7 the influence of aerodynamic forces on the drop formation of a jet of an inviscid liquid is demonstrated. In 8 wave formation, the growth of small wave-like deviations of the jet centerline due to aerodynamic forces is described.

1 Effect of surface tension

We shall investigate an infinitely long steady liquid jet with circular cylindrical cross-section, neglecting the effect of the earth's force of gravity. The radius of the cylinder is a, the surface tension α (force/length). The fluid is then in equilibrium on all sides under a pressure $\frac{\alpha}{a}$; let the surface have small deviations $\overline{\delta}$ from the cylindrical form; $\overline{\delta}$ is an arbitrary function of x and φ, see Figure 3. At every point of the surface there is then a surface pressure $\frac{\alpha}{a} + q_\alpha$ is produced, where q_α is the local deviation from the pressure previously considered uniform. Then we have:

$$\frac{\alpha}{a} - q_\alpha = \alpha \left(\frac{1}{R_1} + \frac{1}{R_2} \right).$$

Here $\frac{1}{R_1}$ and $\frac{1}{R_2}$ are the curvatures of the two planes which are perpendicular to each other. Let us consider the deviations $\overline{\delta}$ and their derivatives with respect to x and φ so small that product terms can be neglected in the calculation of $\frac{1}{R_1}$ and $\frac{1}{R_2}$ and that the direction of the surface pressure can p. 137
be assumed to be perpendicular to the original cylinder surface. We shall determine $\frac{1}{R_1}$ and $\frac{1}{R_2}$ for a cross-section and for a longitudinal section:

$$\frac{1}{R_1} = \frac{r^2 + 2\left(\frac{\partial r}{\partial \varphi}\right)^2 - r\,\frac{\partial^2 r}{\partial \varphi^2}}{\left[r^2 + \left(\frac{\partial r}{\partial \varphi}\right)^2\right]^{3/2}}.$$

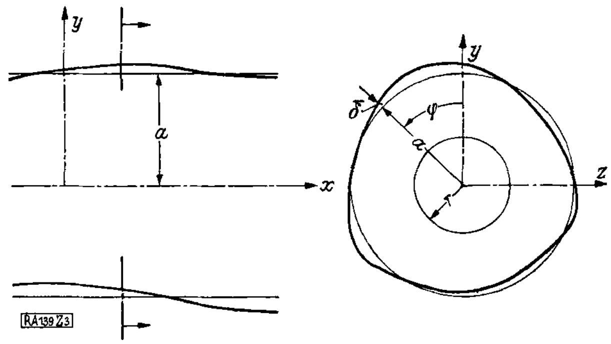

Figure 3: Notations in the jet for arbitrary surface deviations.

With

$$r = a + \overline{\delta}; \frac{\partial r}{\partial \varphi} = \frac{\partial \overline{\delta}}{\partial \varphi}, \text{and so on, we get:}$$

$$\frac{1}{R_1} = \frac{1}{a} - \frac{\overline{\delta}}{a^2} - \frac{1}{a^2}\frac{\partial^2 \overline{\delta}}{\partial \varphi^2},$$

and likewise

$$\frac{1}{R_1} = -\frac{\partial^2 \overline{\delta}}{\partial x^2}.$$

Finally, we have

$$q_\alpha = -a\left(\frac{1}{a^2} + \frac{1}{a^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial x^2}\right)\overline{\delta}. \tag{1}$$

The subsequent investigation will use the Navier-Stokes equations, neglecting the product terms of the velocity and its derivatives as small quantities of higher order. Only linear equations are obtained, so that the complete solution can be formed by superposition of particular solutions. Consequently the external deviations can also be produced by superposition of individual deviations. If they are expanded in a Fourier series according to φ, then every term is to be investigated individually. For the breakup of the jet, the only deviations of importance are rotation-symmetric deviations, in which $\overline{\delta}$ is independent of φ, and one-sided deviations, which correspond to cosine terms. In the latter, the circular cross-section does not change, but is only shifted to the side, whereby the displacement is a function of x.

In rotation-symmetric deviations we have

$$q_\alpha = -\alpha \left(\frac{1}{a^2} + \frac{\partial^2}{\partial x^2} \right) \overline{\delta}. \tag{1a}$$

If we expand the deviation into a Fourier series with respect to x and investigate the term with wavelength $l = 2\pi \frac{a}{\zeta}$, where ζ is the ratio of the circumference of the jet to the wavelength, we get:

$$\overline{\delta} = \overline{\delta}^* \cos \frac{2\pi x}{l} = \overline{\delta}^* \cos \zeta \frac{x}{a}$$

and

$$q_\alpha = -\frac{\alpha \overline{\delta}}{a^2}(1 - \zeta^2).$$

For $\zeta < 1$ corresponding to $l > 2\pi a$, we get for positive $\overline{\delta}$ negative additional pressures. The forces due to the surface tension tend to increase the deviations (unstable condition). For $\zeta > 1$, $l < 2\pi a$, the surface forces tend to suppress the deviations; for $\zeta = 1$, $l = 2\pi a$, the surface forces are equal to zero. We thus see that in consequence of surface tension alone only wave-shaped rotation-symmetric disturbances with $l > 2\pi a$ can cause the jet to break up.

2 Exact solution for rotation-symmetric disturbances

In Figure 4, x and r are cylindrical coordinates, and u and v are axial and radial velocities. Δ and Δ_1 are the following symbols:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} = \frac{\partial^2 f}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} f \right),$$
$$\Delta f_1 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{f}{r^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rf) \right).$$

The Navier-Stokes equations now read: p. 138

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \eta \Delta u, \tag{2}$$
$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial r} + \eta \Delta_1 v. \tag{3}$$

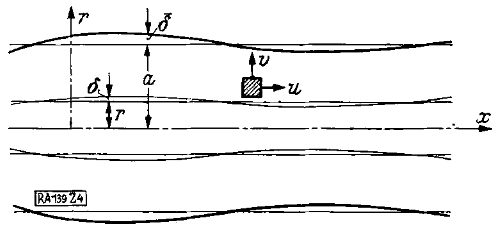

Figure 4: Notation for rotation-symmetric deviations for derivation of the exact equations.

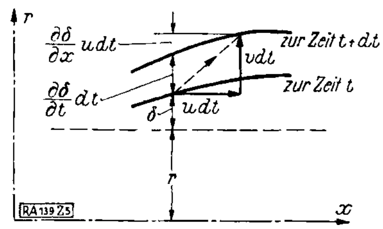

Figure 5: Relation between v and δ.
Translator's note: *zur Zeit* = *at time*.

The equation of continuity gives

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial (rv)}{\partial r} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0. \tag{4}$$

If we differentiate (2) and carry out with the terms of (3) the operation

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\,r\right) = \frac{\partial}{\partial r} + \frac{1}{r}$$

and add the equations thus transformed, the terms in u and v drop out on account of (4); there remains

$$\Delta p = 0. \tag{5}$$

If equation (3) is subjected to the operation Δ_1, p drops out in consequence of (5) and we get the differential equation for v:

$$\rho\,\frac{\partial}{\partial t}\,\Delta_1 v = \eta \Delta_1^2 v. \tag{6}$$

The fluid particles which in the undisturbed jet lie on the same cylindrical surface of radius r form in consequence of the disturbances of a rotation-symmetric surface with the meridian line $r + \delta$.

Here δ is a function of x, r, and the time t. After a time $\mathrm{d}t$ we get the meridian line

$$r + \delta + \frac{\partial \delta}{\partial t}\,\mathrm{d}t.$$

A fluid particle has thus moved radially by an amount $v\,\mathrm{d}t$, lengthwise by an amount $u\,\mathrm{d}t$, and therefore corresponds to a different x. We get

$$\frac{\partial \delta}{\partial t}\,\mathrm{d}t = v\,\mathrm{d}t - \frac{\partial \delta}{\partial x}\,u\,\mathrm{d}t. \quad \text{(Figure 5)}$$

If we neglect product terms we get

$$v = \frac{\partial \delta}{\partial t}\,. \tag{7}$$

Substituting into (6), we get a differential equation for δ:

$$\frac{\partial}{\partial t}\,\Delta_1 \left(\Delta_1 - \frac{\rho}{\eta}\,\frac{\partial}{\partial t}\right)\delta = 0. \tag{8}$$

The order of operations is arbitrary and the equation falls apart into

$$\Delta_1 \delta_1 = 0, \tag{8a}$$

and

$$\left(\Delta_1 - \frac{\rho}{\eta}\,\frac{\partial}{\partial t}\right)\delta_2 = 0. \tag{8b}$$

The other possible solution $\dfrac{\partial \delta}{\partial t} = 0$ is possible but of no importance, since it returns a situation without motion. p. 139

If the solutions δ_1 of (8a) and δ_2 of (8b) are known, the notation of (8) becomes

$$\delta = \delta_1 + \delta_2.$$

If from (7) we substitute

$$v = \frac{\partial \delta}{\partial t} = \frac{\partial \delta_1}{\partial t} + \frac{\partial \delta_2}{\partial t}$$

into equation (3), $\dfrac{\partial p}{\partial r}$ may be expressed in terms of δ_1 and δ_2:

$$\frac{\partial p}{\partial t} = \eta\,\frac{\partial}{\partial t}\left[-\frac{\rho}{\eta}\,\frac{\partial \delta_1}{\partial t} + \Delta_1 \delta_1 - \frac{\rho}{\eta}\,\frac{\partial \delta_2}{\partial t} + \Delta_1 \delta_2\right].$$

On account of equations (8a) and (8b) all terms after the first in the brackets vanish, and

$$p = -\rho \frac{\partial^2}{\partial t^2} \int \delta_1 \, \mathrm{d}r. \tag{9}$$

In carrying out the integration we must take account of the relation (5): $\Delta p = 0$, namely, together with (9):

$$\Delta \int \delta_1 \, \mathrm{d}r = 0.$$

The boundary conditions for $r = a$ are the following:

1. The displacement $\delta = \delta_1 + \delta_2$ is, on the boundary, equal to to the boundary displacement $\overline{\delta}$.
2. The shear stress τ becomes at the boundary $\overline{\tau} = 0$.
3. The boundary pressure p is connected with the boundary loading q_α given in terms of δ. Check

Condition (1) gives

$$(\delta_1 + \delta_2)_{r=a} = \overline{\delta}. \tag{10}$$

Condition (2) gives

$$\overline{\tau} = \eta \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x} \right)_{r=a} = 0.$$

This equation is differentiated with respect to x and subsequently u then eliminated by the use of the continuity equation (4). If we then replace v by $\frac{\partial \delta}{\partial t}$, we get

$$\left[\left(\Delta_1 - 2 \frac{\partial^2}{\partial x^2} \right) \delta \right]_{r=a} = 0.$$

If we put in $\delta = \delta_1 + \delta_2$, and take account of equations (8a) and (8b), we get

$$\left[\frac{\rho}{\eta} \frac{\partial \delta_2}{\partial t} - 2 \frac{\partial^2(\delta_1 + \delta_2)}{\partial x^2} \right]_{r=a} = 0. \tag{11}$$

Condition (3) for the boundary pressure gives

$$\left(p + 2\eta \frac{\partial v}{\partial r} \right)_{r=a} = q_\alpha.$$

With q_α, p, and v, from equations (1a), (9), and (7), respectively, we get

$$\left[\rho \frac{\partial^2}{\partial t^2} \int \delta_1 \, \mathrm{d}r + 2\eta \frac{\partial^2(\delta_1 + \delta_2)}{\partial t \partial r}\right]_{r=a} = \alpha \left(\frac{1}{a^2} + \frac{\partial^2}{\partial x^2}\right) \overline{\delta}, \tag{12}$$

with $\Delta \int \delta_1 \, \mathrm{d}r = 0$ as a condition on the integration.

Equations (8a) and (8b) are the differential equations for δ_1 and δ_2; equations (10) to (12) are the boundary conditions.

We turn now from the general solution to the particular solutions.

For δ we make the substitution

$$\delta = \delta^* e^{\mu t} \cos \zeta \frac{x}{a}. \tag{13}$$

Similar substitutions hold for δ_1, and δ_2, $\overline{\delta}$, and p.

From such particular solutions practically any general solution can be set up. δ^* is only a function p. 140
of r, which for $r = a$ takes on the value $\overline{\delta}^*$. The meridian line for each particular solution is a cosine curve with the wavelength $l = 2\pi \frac{a}{\zeta}$.

Equations (8a) and (8b) give

$$\left(-\frac{\zeta^2}{a^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) \delta_1^* = 0 \tag{14a}$$

$$\left(-\frac{\zeta^2}{a^2} - \frac{\mu\rho}{\eta} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) \delta_2^* = 0 \tag{14b}$$

The parenthetical expressions in these equations differ only in the terms independent of r. In (14a) we have $-\frac{\zeta^2}{a^2}$, in (14b) $-\frac{\zeta^2}{a^2} - \frac{\mu\rho}{\eta}$.

For the sake of uniformity, we put $-\frac{\zeta^2}{a^2} - \frac{\mu\rho}{\eta} = -\frac{\zeta_1^2}{a^2}$, or

$$\zeta_1^2 = \zeta^2 + \frac{\mu\rho a^2}{\eta}. \tag{15}$$

The solutions are Bessel functions of the first order and indeed of the first kind since $\delta = 0$ for

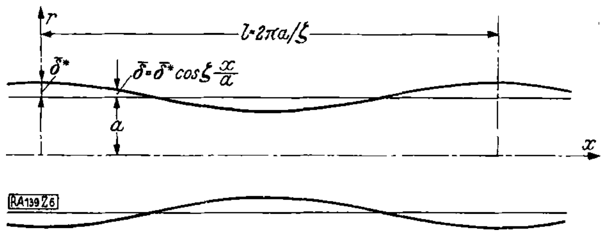

Figure 6: Rotation-symmetric disturbance as a cosine wave with length $2\pi a/\zeta$.

$r = 0$, with arguments $\zeta \frac{r}{a}$ and $\zeta_1 \frac{r}{a}$, respectively. Their series developments are

$$\frac{1}{2}\left(\zeta \frac{r}{a}\right) + \frac{1}{2 \cdot 2 \cdot 4}\left(\zeta \frac{r}{a}\right)^3 + \frac{1}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6}\left(\zeta \frac{r}{a}\right)^5 + \cdots = F_1\left(\zeta \frac{r}{a}\right)$$

and/or

$$\frac{1}{2}\left(\zeta_1 \frac{r}{a}\right) + \frac{1}{2 \cdot 2 \cdot 4}\left(\zeta_1 \frac{r}{a}\right)^3 + \frac{1}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6}\left(\zeta_1 \frac{r}{a}\right)^5 + \cdots = F_1\left(\zeta_1 \frac{r}{a}\right).$$

In the notation of Bessel functions we have

$$F_1\left(\zeta \frac{r}{a}\right) = -iJ_1\left(i\zeta \frac{r}{a}\right).$$

In order to obtain δ_1 and δ_2, these functions can further be multiplied by magnitudes independent of r. These are immediately determined in such a manner that equation (10) $(\delta_1 + \delta_2)_{r=a} = \overline{\delta}$ is satisfied:

$$\delta_1^* = (1 - c)\overline{\delta}^* F_1\left(\zeta \frac{r}{a}\right) \Big/ F_1(\zeta), \tag{16a}$$

$$\delta_2^* = \qquad c\overline{\delta}^* F_1\left(\zeta_1 \frac{r}{a}\right) \Big/ F_1(\zeta_1). \tag{16b}$$

To determine c, $\delta_1 = \delta_1^* e^{\mu t} \cos \zeta \frac{x}{a}$ and a corresponding expression for δ_2 are substituted in equation (11). The F_1 functions cancel, since $r = a$, and the equation is further divided through by $\overline{\delta}^* e^{\mu t} \cos \zeta \frac{r}{a}$, giving

$$\mu \frac{\rho}{\eta} c + 2\frac{\zeta^2}{a^2} = 0.$$

From this p. 141

$$c = -\frac{2\zeta^2\eta}{\mu\rho a^2}. \tag{17}$$

With this value of c the solutions δ_1 and δ_2 must satisfy the last boundary condition (12). In this we have

$$\int \delta_1 \, \mathrm{d}r = \frac{(1-c)\overline{\delta}^* a}{\zeta F_1(\zeta)} e^{\mu t} \cos\zeta\frac{x}{a} \cdot \int F_1\left(\zeta\frac{r}{a}\right) \mathrm{d}\left(\zeta\frac{r}{a}\right).$$

The integral gives the Bessel function of order zero:

$$\int F_1\left(\zeta\frac{r}{a}\right) \mathrm{d}\left(\zeta\frac{r}{a}\right) = 1 + \frac{1}{2\cdot 2}\left(\zeta\frac{r}{a}\right)^2 + \frac{1}{2\cdot 2\cdot 4\cdot 4}\left(\zeta\frac{r}{a}\right)^4 + \cdots = F_0\left(\zeta\frac{r}{a}\right). \tag{18}$$

The integration condition $\Delta \int \delta_1 \, \mathrm{d}r = 0$ is thus satisfied. With this evaluation of the integral, equation (12) gives, if $F_1'(\zeta) = \dfrac{\mathrm{d}F_1(\zeta)}{\mathrm{d}\zeta}$:

$$\rho\mu^2\left(1 + \frac{2\zeta^2\eta}{\mu\rho a^2}\right)\frac{a}{\zeta}\frac{F_0(\zeta)}{F_1(\zeta)} + 2\eta\mu\left[\left(1 + \frac{2\zeta^2\eta}{\mu\rho a^2}\right)\frac{\zeta}{a}\frac{F_1'(\zeta)}{F_1(\zeta)} - \frac{2\zeta^2\eta}{\mu\rho a^2}\frac{\zeta_1 F_1'(\zeta_1)}{aF_1(\zeta_1)}\right] = \frac{\alpha}{a^2}(1-\zeta^2). \tag{19}$$

After transformation and utilization of the relations

$$F_1'(\zeta) = F_0(\zeta) - \zeta^{-1}F_1(\zeta)$$

and

$$\frac{\eta}{\mu\rho a^2} = \frac{1}{\zeta_1^2 - \zeta^2},$$

we have

$$\mu^2\frac{F_0(\zeta)}{2\zeta^{-1}F_1(\zeta)} + \mu\frac{\eta\zeta^2}{\rho a^2}\left[2\frac{F_0(\zeta)}{\zeta^{-1}F_1(\zeta)} - 1 \right.$$
$$\left. + \frac{2\zeta^2}{(\zeta_1^2-\zeta^2)}\left(\frac{F_0(\zeta)}{\zeta^{-1}F_1(\zeta)} - \frac{F_0(\zeta_1)}{\zeta_1^{-1}F_1(\zeta_1)}\right)\right] = \frac{\alpha}{2\rho a^3}(1-\zeta^2)\zeta^2. \tag{20}$$ [3]

Thus the equation for μ for a given ζ is set up.

[3] The corresponding equation, without consideration of viscosity, was given by Lord Rayleigh, Proc. London Math. Soc. 1872.

The coefficient of μ^2 is

$$\frac{F_0(\zeta)}{2\zeta^{-1}F_1(\zeta)} = \frac{1 + \dfrac{\zeta^2}{4} + \dfrac{\zeta^4}{64} + \dfrac{\zeta^6}{2304} + \dfrac{\zeta^8}{147456} + \dots}{1 + \dfrac{\zeta^2}{8} + \dfrac{\zeta^4}{192} + \dfrac{\zeta^6}{9216} + \dfrac{\zeta^8}{737280} + \dots}$$
$$= 1 + \frac{\zeta^2}{8} - \frac{\zeta^4}{192} + \frac{\zeta^6}{3072} - \frac{\zeta^4}{46080} + \dots,$$

and it differs only slightly from unity for $\zeta < 1$.

The square brackets contain the same expression and give the series development

$$4\left(1 + \frac{\zeta^2}{8} - \frac{\zeta^4}{192} + \frac{\zeta^6}{3072} - \frac{\zeta^8}{46080} + \dots\right) - 1$$
$$+ \frac{4\zeta^2}{\zeta_1^2 - \zeta^2}\left(-\frac{\zeta_1^2 - \zeta^2}{8} + \frac{\zeta_1^4 - \zeta^4}{192} - \frac{\zeta_1^6 - \zeta^6}{3072} + \frac{\zeta_1^8 - \zeta^8}{46080} - \dots\right)$$
$$= 3 + \frac{1}{48}\zeta^2\zeta_1^2 - \frac{1}{768}\zeta^2\zeta_1^2(\zeta^2 + \zeta_1^2) + \frac{1}{11520}\zeta^2\zeta_1^2(\zeta^4 + \zeta^2\zeta_1^2 + \zeta_1^4) + \dots .$$

This series likewise differs little from 3 for $\zeta < 1$. In place of equation (20) we can take with great accuracy the quadratic equation in μ:

$$\mu^2 + \mu\frac{3\eta}{\rho a^2}\zeta^2 = \frac{\alpha}{2\rho a^3}(1 - \zeta^2)\zeta^2. \tag{21}$$

If we have determined μ from this equation, we can improve the coefficients of μ^2 and μ in (20) and get thereby a new quadratic equation, which gives, however, practically the same μ solutions.

In 3 the approximate equation (21) will be derived directly in intuitive fashion, whereby the p. 142
physical significance of the approximation will appear.

If the right-hand side of equation (21) is equal to zero, that is, if $\zeta = 0$ or 1, one value of μ is 0, the other negative. If $0 < \zeta < 1$ the right-hand side is positive and one value is positive, the other negative. Only these positive values of μ are important. The negative μ values correspond to decreasing disturbances; the pure imaginary μ values correspond to oscillations, and the imaginary values correspond to negative real parts damped oscillations.

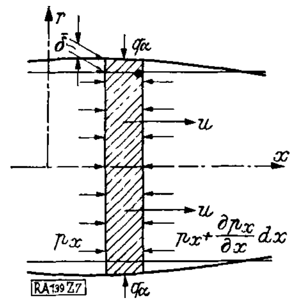

Figure 7: Notations for rotation-symmetric deviations for derivation of the approximate equations.

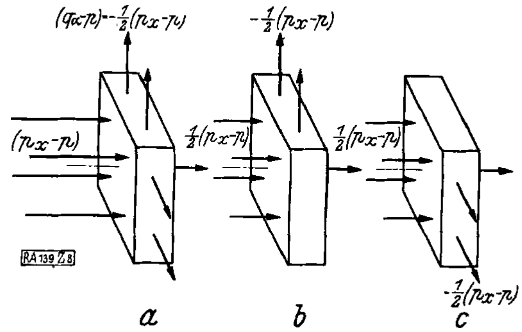

Figure 8: Resolution of the tension condition.

3 Elementary derivation of the equation for rotation-symmetric disturbances

Only disturbances of long wavelength come into equation (Figure 7). If velocity v is small, u and p are practically functions of x only. The problem can be treated in one dimension.

To the exterior pressure $q_\alpha = -\alpha \left(\frac{1}{a^2} + \frac{\partial^2}{\partial x^2} \right) \bar{\delta}$ there corresponds in each cross-section a unit interior pressure p and a unit pressure p_x in the x direction. We consider a disk-shaped portion of the jet of length dx. The radius is $a + \bar{\delta}$. After a time dt this portion is displaced to the right by an amount $u\,\mathrm{d}t$. The new radius will be

$$a + \bar{\delta} + \frac{\partial \bar{\delta}}{\partial t}\,\mathrm{d}t + \frac{\partial \bar{\delta}}{\partial x}\,u\,\mathrm{d}t.$$

The last term is a small quantity of higher order and may be neglected. In determining the variation with time it is sufficient to consider the variation for a fixed x, while the variation resulting from the axial motion of the element is unimportant. These statements also hold for the time variation of p, p_x, and u.

The mean value of the pressure in the x, y, and z directions, p_x, q_α, and q_α[4] gives for the internal pressure

$$p = \frac{1}{3}(p_x + 2q_\alpha).$$

Influence of viscosity. The connection between the internal tensions $\sigma_x = -p_x$, $\sigma_y = -q_\alpha$ and τ_{xy} with the velocities follows from the similarity in form to the elasticity problem. Poisson's number becomes $\nu = 0.5$; in place of the shear modulus G we have η, and in place of the modulus of elasticity $E = 2G(1 + \nu)$ we have 3η. In place of displacements we are to take u and v.

For the shear stresses we have

$$\tau_{xy} = \eta \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right);$$

u is independent of y, v is intrinsically small and, in consequence of the long waves, varies only slightly with x. Consequently τ_{xy} becomes meaningless for us, a fact which agrees with one boundary condition.

Of the normal tensions, only $p_x - p$ in the x direction, $q_\alpha - p = -\frac{1}{2}(p_x - p)$ in the y and z directions affect the shape of the element (Figure 8a). The mean pressure p is without effect on p. 143
incompressible materials. If the tension is divided according to Figs. 8b and 8c, we get for the tension according to 8b

$$\frac{\partial u_1}{\partial x} = -\frac{\partial v}{\partial y}; \quad E\frac{\partial u_1}{\partial x} = \sigma_x - \nu\sigma_y \quad \text{(for the elastic problem)}$$
$$3\eta\frac{\partial u_1}{\partial x} = -\frac{1}{2}(p_x - p) - \frac{1}{2} \cdot \frac{1}{2}(p_x - p),$$

or

$$4\eta\frac{\partial u_1}{\partial x} = p - p_x.$$

The same holds for the tension according to Figure 8c with the velocity u_z.

With $u = u_1 + u_2 = 2u_2$, and $p = \frac{1}{3}(p_x + 2q_\alpha)$: $q_\alpha - p_x = 3\eta \dfrac{\partial u}{\partial x}$ and with the value of q_α, we

[4] Editor's note: The original writes q twice. The Boulder translation used q_α instead, which I assume was intended by Weber. I assume that the writing of q_α twice was meant to indicate it is being included twice, hence the coefficient of 2, as this is confusingly written.

obtain

$$p_x = -\alpha \left(\frac{1}{a^2} + \frac{\partial^2}{\partial x^2} \right) \overline{\delta} - 3\eta \frac{\partial u}{\partial x}. \tag{22}$$

Continuity equation. The disk of radius $a + \overline{\delta}$ and length dx has the volume $\pi(a + \overline{\delta})^2 \, \mathrm{d}x$; this changes in the time dt, since the face surfaces have the velocities u and $u + \frac{\partial u}{\partial x} \mathrm{d}x$ by an amount

$$\frac{\partial V}{\partial t} \mathrm{d}t = \frac{\partial}{\partial t} \pi(a + \overline{\delta})^2 \, \mathrm{d}t \, \mathrm{d}x + \pi(a + \overline{\delta})^2 \frac{\partial u}{\partial x} \mathrm{d}x \, \mathrm{d}t.$$

Since the fluid is not compressible, $\frac{\partial V}{\partial t} = 0$, and we get by neglect of higher terms

$$\frac{2}{a} \frac{\partial \delta}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{23}$$

Equation of motion. The pressures p_x and $p_x + \frac{\partial p_x}{\partial x} \mathrm{d}x$ in the faces of the cylindrical disk accelerate it in the x direction:

$$-\frac{\partial}{\partial x} [\pi(a + \overline{\delta})^2 p_x] \, \mathrm{d}x = \pi(a + \overline{\delta})^2 \, \mathrm{d}x \rho \frac{\partial u}{\partial t},$$

or

$$-\frac{\partial p_x}{\partial x} = \rho \frac{\partial u}{\partial t}. \tag{24}$$

Differential equation for $\overline{\delta}$. If in equation (24) we put p_x from (22) and u from (23), we get

$$\frac{\partial^2 \overline{\delta}}{\partial t^2} - \frac{3\eta}{\rho} \frac{\partial^2 \overline{\delta}}{\partial t \partial x} = -\frac{\alpha}{2\rho a} \frac{\partial^2}{\partial x^2} \left(\overline{\delta} + a^2 \frac{\partial^2 \overline{\delta}}{\partial x^2} \right). \tag{25}$$

With the earlier substitution $\overline{\delta} = \overline{\delta}^* e^{\mu t} \cos \zeta \frac{x}{a}$ we get again

$$\mu^2 + \mu \frac{3\eta}{\rho a^2} \zeta^2 = \frac{\alpha}{2\rho a} (1 - \zeta^2) \zeta^2. \tag{21}$$

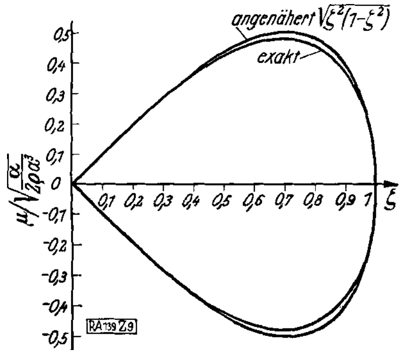

Figure 9: Dependence of the quantity $\mu \Big/ \sqrt{\dfrac{\alpha}{2\rho a^3}}$ on ζ, exactly and approximately, for inviscid fluids. Translator's note: *angenähert* = *approx. exakt* = *exact*.

4 Drop formation in an inviscid fluid

The deviation $\overline{\delta}$ of the surface consists of a superposition of different cosine (or sine) functions. The equation for the coefficient of t in the exponential function is, exactly,

$$\mu^2 = \frac{\alpha}{2\rho a^3}(1-\zeta^2)\zeta^2 \frac{2\zeta^{-1}F_1(\zeta)}{F_0(\zeta)} \tag{26}$$

and approximately

$$\mu^2 = \frac{\alpha}{2\rho a^3}(1-\zeta^2)\zeta^2. \tag{27}$$

Figure 9 shows the dependence of the quantity $\mu \Big/ \sqrt{\dfrac{\alpha}{2\rho a^3}}$ for both equations. The difference is p. 144
insignificant. For a certain $\zeta = \zeta_{\text{opt}}$ we get for μ the maximum value μ_{opt}. For the approximate solution we have

$$\zeta_{\text{opt}} = \sqrt{0.5} \quad \text{and} \quad \mu_{\text{opt}} = \sqrt{\frac{\alpha}{8\rho a^3}}. \tag{28}$$

For the exact solution ζ_{opt} is only slightly greater and μ_{opt} is 3% smaller. The difference is so small that from now on only the approximate solution will be calculated.

If for $t = 0$ and all ζ values the initial disturbances are equally strong, the wavelengths which correspond to ζ_{opt} and neighboring values will increase more strongly and after some time will be

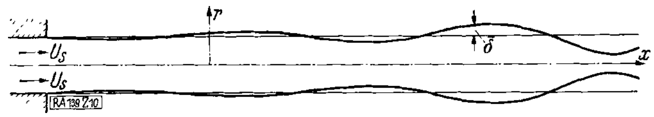

Figure 10: Jet from a nozzle with rotation-symmetric disturbance.

barely noticeable. The ratio of the wavelength to be observed to the jet diameter is, since $\dfrac{2\pi a}{l}$,

$$\frac{l_{\text{opt}}}{2a} = \frac{\pi}{\zeta_{\text{opt}}} = \pi\sqrt{2} = 4.44.$$

Besides the positive value of μ we get for each ζ a negative value of equal magnitude; the corresponding disturbance disappears gradually. By superposition we get the disturbances $\overline{\delta}^* \cosh \mu t \cos \zeta \dfrac{x}{a}$ and $\overline{\delta}^* \sinh \mu t \sin \zeta \dfrac{x}{a}$; for $t = 0$ these give an initial disturbance without radial velocity, or radial deviation, respectively. In the following, we shall consider only the disturbances with positive μ.

If the jet leaves the nozzle with velocity U_S, then it acquires initial disturbances at the nozzle, which are carried forward with the jet and at the same time grow, Figure 10.

Let the coordinate system be fixed to the jet and move to the right with velocity U_S. For the nozzle, $x = -U_S t$.

Let the initial disturbance at the nozzle be a cosine function of the time; if the disturbance in the jet does not vary with the time, we should have

$$\overline{\delta} = \overline{\delta}^* \cos \zeta \frac{x}{a}.$$

Since there is a time variation, we have approximately[5]

$$\overline{\delta} = \overline{\delta}^* e^{\mu(t + x/U_S)} \cdot \cos \zeta \frac{x}{a}. \tag{30}$$

For the nozzle, that is, for $x = -U_S t$, we have $\overline{\delta}_{\text{nozzle}} = \overline{\delta}^* \cos \dfrac{U_S t}{a}$.

The solution is only an approximation which holds for sufficiently large U_{S}. In order to test its accuracy, let us compare the approximation with an exact solution of differential equation (25). For inviscid fluids the latter is

$$\frac{\partial^2 \overline{\delta}}{\partial t^2} = -\frac{\alpha}{2\rho a}\left(\frac{\partial^2 \overline{\delta}}{\partial x^2} + a^2 \frac{\partial^4 \overline{\delta}}{\partial x^4}\right). \tag{31}$$

[5]Editor's note: There is no equation (29) in the original.

We put

$$\overline{\delta} = \overline{\delta}^* e^{\mu_\mathrm{S}(t+x/U_\mathrm{S})} \cdot \cos\zeta \frac{x - U_1 t}{a} = \overline{\delta}^* \mathrm{Re}\, e^{\left(\frac{\mu_\mathrm{S}}{U_\mathrm{S}} + i\frac{\zeta}{a}\right)x + \left(\mu_\mathrm{S} - i\zeta\frac{U_1}{a}\right)t \ldots}. \tag{32}$$

At any certain time the disturbance is represented as before by a forced cosine curve. For a certain ζ of the cosine curve the actual value μ_S is sought for the jet velocity U_S. The cosine curve may at the same time move slowly with the velocity U_1 relative to the jet. p. 145

We assume ζ/a and $\dfrac{\mu_\mathrm{S}}{U_\mathrm{S}} = \dfrac{\mu}{U}$ as given, i.e., we take μ from equation (27) and a velocity U, and only later determine U_S, which will differ from U as little as μ_S does from μ. If we make the substitution in equation (31), we get for the determination of μ_S and $\zeta\dfrac{U_1}{a} = \nu_\mathrm{S}$ the two equations:

$$\mu_\mathrm{S}^2 - \nu_\mathrm{S}^2 = \frac{a}{2\rho a^3}\left[\zeta^2 - \left(\frac{\mu}{U}a\right)^2 - \zeta^4 + 6\zeta^2\left(\frac{\mu}{U}a\right)^2 - \left(\frac{\mu}{U}a\right)^4\right]$$

$$\mu_\mathrm{S}\nu_\mathrm{S} = \frac{a}{2\rho a^3}\left[\zeta\left(\frac{\mu}{U}a\right) + 2\left(-\zeta^3\left(\frac{\mu}{U}a\right) + \zeta\left(\frac{\mu}{U}a\right)^3\right)\right].$$ [6]

If we put

$$\frac{1}{U}\sqrt{\frac{\alpha}{\rho a}} = \varepsilon,$$

we get, with

$$\mu = \sqrt{\frac{\alpha}{2\rho a^3}}\sqrt{\zeta^2(1-\zeta^2)},$$

$$\left.\begin{aligned}\mu_\mathrm{S}^2 - \nu_\mathrm{S}^2 &= \mu^2\left[1 + \frac{1}{2}(6\zeta^2 - 1)\varepsilon^2 + \zeta^2(\zeta^2 - 1)\varepsilon^4\right],\\ \mu_\mathrm{S}\nu_\mathrm{S} &= \frac{\mu^2}{\sqrt{2(1-\zeta^2)}}[(1-2\zeta^2)\varepsilon + \zeta^2(1-\zeta^2)\varepsilon^3].\end{aligned}\right\} \tag{33}$$ [7]

If $\varepsilon < 1$, then ν_S will also be small in comparison with μ_S and $\mu_\mathrm{S} \approx \mu$.

Consequently, the condition for the validity of solution (30) is: Check

$$U_\mathrm{S} > \sqrt{\frac{\alpha}{\rho a}},$$

[6] Editor's note: The 1948 translation changed the sign from − to + in this equation. I am assuming this is a correction, but I have not checked.

[7] Editor's note: Again, the 1948 translation changed the sign from − to + in this equation. I again assume this is a correction, but I have not checked.

since then we will also have $U \approx U_\mathrm{S}$.

Numerical example.[8] Water jet, nozzle diameter 0.5 mm, $a = 0.025$ cm, $\rho = 0.00102$ g cm^{-4} sec^2, $\alpha = 0.072$ g cm^{-1}. For $\zeta = 0.8$, we have $\mu = 721$ s^{-1}; $\sqrt{\dfrac{\alpha}{\rho a}} = 53$ cm^{-1}.

For $U = 200$ cm sec^{-1}, we get $\varepsilon = 0.266$.

Equations (33) result in:

$$\mu_\mathrm{S}^2 - \nu_\mathrm{S}^2 = 1.10\mu^2,$$
$$\mu_\mathrm{S}\nu_\mathrm{S} = -0.0826\mu^2,$$
$$\mu_\mathrm{S} = 1.05\mu; \quad \nu_\mathrm{S} = -0.0785\mu = -56.6\ \mathrm{s}^{-1}; \quad U_1 = \frac{\nu_\mathrm{S} a}{\zeta} = 1.77\ \mathrm{cm}^{-1}.$$

μ_S is slightly greater than μ, and the velocity of the cosine curve relative to the jet is insignificant in comparison with the jet velocity.

In the following it will be supposed that U_S is so large that equation (30) holds for the nozzle jet.

Breakup time and breakup length. The jet from the nozzle shows no visible disturbances near the nozzle; only after some distance does wave formation set in, and the constrictions grow and bring Check the jet to form drops. The distance from the nozzle to the point of drop formation is the breakup length L, and to it corresponds the breakup time $T = \dfrac{L}{U_\mathrm{S}}$.

Critical for the breakup are the wavelengths which correspond to ζ_opt, or to ζ values only slightly different. We replace the complete disturbances at the nozzle with an equivalent disturbance $\overline{\delta}^* \cos \zeta_\mathrm{opt} \dfrac{U_\mathrm{S} t}{a}$ and demand that the observed breakup length shall be present. It may be assumed that the exponential law is valid up to the breakup point since the section of the jet, in which the deviations $\overline{\delta}$ are not small, is short.

We get p. 146

$$\overline{\delta}^* e^{T\mu_\mathrm{opt}} = a,$$

or

$$T = \frac{1}{\mu_\mathrm{opt}} \ln \frac{a}{\overline{\delta}^*}; \quad L = T\,U_\mathrm{S}.$$

[8] Editor's note: The numbers here are printed as in the 1948 translation, not the 1931 original. I assume that the 1948 translation corrected these numbers.

Experiments with water in the nozzles investigated by A. Haenlein gave the mean value

$$T\mu_{\mathrm{opt}} = 12.$$

It follows that

$$\overline{\delta}^{*} = a/e^{12} = a/160\,000.$$

At this order of magnitude the disturbances in the neighborhood of the nozzle are not noticeable. The initial disturbances at the nozzles are obviously not constantly equal, but are subject to strong variations. In spite of this, the breakup time and, for a constant jet velocity U_{S} also the breakup length, remain almost unchanged. For example, if $\overline{\delta}^{*}$ is reduced 20 fold T and L are reduced only by 25%.

From the scatter plot of the experimental results it is found that up to a certain velocity the Check
breakup length L is proportional to U_{S}. From this we might conclude that the mean value of the initial disturbances is independent of U_{S}. In consequence of the established insensitivity of the length L to changes in $\overline{\delta}^{*}$ another law may also hold. Figure 11 shows the dependence of the length L on U_{S} when $\overline{\delta}^{*}$ varies as U_{S}^{2} and for a value of $2\ \mathrm{m}\cdot\mathrm{s}^{-1}$, $\ln(a/\overline{\delta}^{2}) = 12$. In view of the scatter plot Check
of the experimental results, no decision can be reached.

It remains to be shown how individual discontinuities in the infinite jet grow and spread out. For comparison, let us investigate two cases: the singular constriction shall appear periodically, first at the distances l_{opt}, then at the distance $10 \cdot l_{\mathrm{opt}}$.

In the first case, it is sufficient for $t = 0$ to take the substitution

$$\overline{\delta}_1 = \overline{\delta}_1^{*}\left(\frac{1}{2} + \underline{\cos\frac{x}{\sqrt{2}a}} + \cos\frac{2x}{\sqrt{2}a} + \cos\frac{3x}{\sqrt{2}a} + \dots\right).$$

Only the first, underlined, cosine disturbance will grow; Figure 12a shows the form if the growth of the disturbance continues up to construction according to the exponential law. We have here $T\mu_{\mathrm{opt}} = \ln\frac{a}{\overline{\delta}^{*}}$. According to the magnitude of $\ln a/\overline{\delta}^{*}$, we have different times to reach this condition.

If the disturbances take place at the distance $10 \cdot l_{\mathrm{opt}}$, then

$$\overline{\delta}_{10} = \overline{\delta}_{10}^{*}\left(\frac{1}{2} + \cos\frac{x}{10\sqrt{2}a} + \cos\frac{2x}{10\sqrt{2}a} + \dots + \cos\frac{14x}{10\sqrt{2}a} + \cos\frac{15x}{10\sqrt{2}a} + \dots\right).$$

For sufficiently large times only the underlined terms need to be considered. At time T we get, p. 147

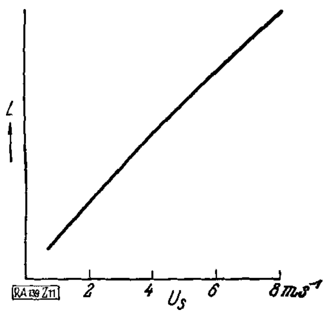

Figure 11: Change of breakup length with U_S in the case where the initial disturbance grows with U_S^2. For $U_S = 2$ m $\cdot$ s^{-1}, $\ln(a/\bar{\delta}^2) = 12$. Check

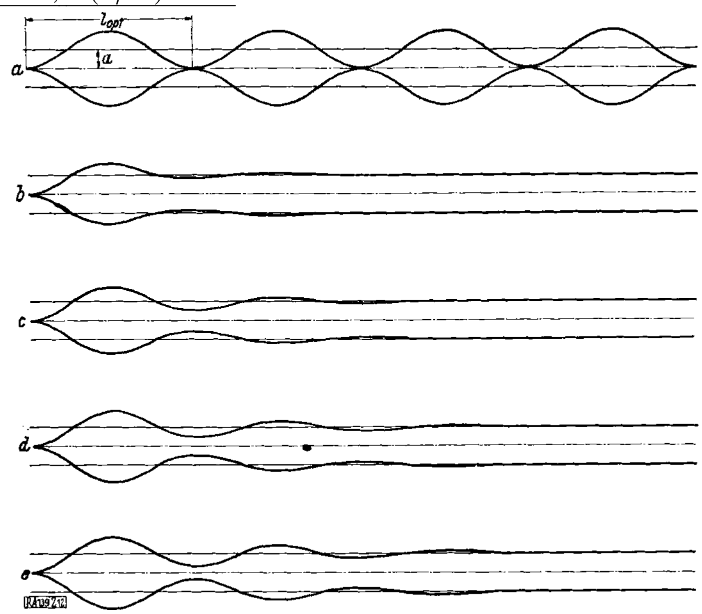

Figure 12: Breakup from singular disturbances a: Disturbances at distance l_{opt}. a to e: Disturbances at distance $10 \cdot l_{opt}$ with $\mu_{opt}T = 3, = 6, = 9$, and $= 12$. The singular disturbances are determined in such a manner that complete constriction occurs. Check

$\mu_{\text{opt}} \cdot T$	$\overline{\delta}_1^*/a$	$\overline{\delta}_{10}^*/a$	$S_1/(\frac{1}{2}al_{\text{opt}})$	$S_{10}/(\frac{1}{2}al_{\text{opt}})$	$\frac{S_1}{S_{10}}$
3	1/20.1	1/143	1/20.1	1/14.3	0.71
6	1/403	1/2060	1/403	1/206	0.51
9	1/8100	1/33850	1/8100	1/3385	0.42
12	1/162700	1/588920	1/162700	1/58892	0.34

Table 1: Numerical values of jet breakup by singular disturbances.

with $\zeta_n = \dfrac{n}{10\sqrt{2}}$

$$\overline{\delta}_{10,t=T} = \overline{\delta}_{10}^* \sum_{n=1}^{n=14} \cos \frac{nx}{10\sqrt{2}a} \cdot e^{2\sqrt{\zeta_n^2(1-\zeta_n^2)}\cdot\mu_{\text{opt}}T}.$$

In Figs. 12b–e the breakup phenomena are sketched for $\mu_{\text{opt}}T = 3, = 6, = 9$, and $= 12$. Here $\overline{\delta}_{10}$ has been determined in such a manner that a complete constriction occurs, i.e., so that

$$\overline{\delta}_{10,t=T} = \overline{\delta}_{10}^* \sum e^{2\sqrt{\zeta_n^2(1-\zeta_n^2)}\cdot\mu_{\text{opt}}T} = a.$$

The magnitude of the singularity is in the first case

$$S_1 = \int_{-0}^{+0} \overline{\delta}_{1,t=0}\,\mathrm{d}x = \frac{1}{2}\overline{\delta}_1^* \cdot l_{\text{opt}};$$

in the second case

$$S_{10} = \int_{-0}^{+0} \overline{\delta}_{10,t=0}\,\mathrm{d}x = \frac{1}{2}\overline{\delta}_1^* \cdot 10\, l_{\text{opt}}.$$

(Integration limits immediately before and after the singular point.)

In Table 1, for the four values of $\mu_{\text{opt}}T$ above, the values

$$\overline{\delta}_1^*/a, \quad \overline{\delta}_{10}^*/a, \quad S_1/\left(\frac{1}{2}al_{\text{opt}}\right), \quad S_{10}/\left(\frac{1}{2}al_{\text{opt}}\right)$$

are presented. The last values show how the magnitudes of the singularities at these two distances must behave, in order that the breakup may occur at the same time. For $\mu_{\text{opt}}T \to \infty$, $S_1/S_{10} \to 0.1$. An isolated strong singular disturbance appearing in the jet produces a sudden earlier breakup, with the point of breakup moving forward with the jet velocity, and with the disturbance propagated laterally[9].

[9]Such a point is shown in Fig. 10b of Haenlein's work.

5 Drop formation in a viscous fluid

The investigation will be carried on with the approximate equation (21), since the difference of the μ values is even less than for inviscid fluids.

We have as a particular solution for the infinite jet

$$\bar{\delta} = \bar{\delta}^* e^{\mu t} \cos \zeta \frac{x}{a},$$

with[10]

$$\mu = -\frac{3}{2}\frac{\eta}{\rho a^2}\zeta^2 \pm \sqrt{\frac{\alpha}{2\rho a^3}(1-\zeta^2)\zeta^2 + \left(\frac{3}{2}\frac{\eta}{\rho a^2}\zeta^2\right)^2}. \tag{35}$$

Figure 13 shows the dependence of $\mu \Big/ \sqrt{\frac{a}{2\rho a^3}}$ on ζ for $\frac{3}{2}\frac{\eta}{\rho a^2} \Big/ \sqrt{\frac{\alpha}{2\rho a^3}} = 0, = 1$ and $= 20$. (The values 1 and 20 correspond to glycerin, $2a = 0.07$ mm, and castor oil, $2a = 0.07$ mm.)

For ζ_{opt} we get the maximum value μ_{opt}. p. 148

We obtain

$$\mu_{\text{opt}} = 1 \Big/ \left(\sqrt{\frac{8\rho}{\alpha}} \cdot a^{1.5} + \frac{6\eta}{\alpha} \cdot a\right), \tag{36}$$

$$\zeta_{\text{opt}} = 1 \Big/ \sqrt{2\left(1 + \sqrt{\frac{9}{2}\frac{\eta^2}{\alpha\rho a}}\right)} \tag{37}$$

and

$$l_{\text{opt}} = 2\pi a \sqrt{2\left(1 + \sqrt{\frac{9}{2}\frac{\eta^2}{\alpha\rho a}}\right)}. \tag{38}$$

If we assume as an approximation that at the nozzle for every fluid only disturbances of equal strength (hence with equal $\bar{\delta}^*$) with ζ_{opt} for the time being, then the values μ_{opt} afford the possibility for formulating the breakup time T. As in 4 we get Check

$$T = \frac{1}{\mu_{\text{opt}}} \ln \frac{a}{\bar{\delta}^*}$$

[10]Editor's note: There is no equation (34) in the original.

and with $\ln(a/\overline{\delta}^*)$ = constant,

$$T = \ln \frac{a}{\overline{\delta}^*} \cdot \left(\sqrt{\frac{8\rho}{\alpha}} \cdot a^{1.5} + \frac{6\eta}{\alpha} \cdot a \right)$$

or, in dimensionless form

$$T \cdot \frac{\alpha^2 \rho}{27\eta^3} = \ln \frac{a}{\overline{\delta}^*} \cdot \left[\left(\frac{2}{9} \frac{\alpha \rho a}{\eta^2} \right)^{3/2} + \left(\frac{2}{9} \frac{\alpha \rho a}{\eta^2} \right) \right]$$

In Figure 14 $T \cdot \dfrac{\alpha^2 \rho}{27\eta^3}$ is represented in logarithmic units as a function of $\dfrac{2}{9} \dfrac{\alpha \rho a}{\eta^2}$. We have taken $\ln \dfrac{a}{\overline{\delta}^*} = 12$ here. The experimental results of A. Haenlein are included for comparison. Now a slightly viscous fluid the first term is dominant and T is proportional to $a^{1.5}$; for very viscous fluids it is the second term, and T is proportional to a.

6 Surface stress due to air effects

Check

Let an essentially cylindrical rigid body of radius a and minor surface deviations $\overline{\delta}$, Figure 15, move through quiescent air with the velocity U_S. The coordinate system x, y, z and x, r, φ, respectively, is connected with the air. The air particles acquire small velocities u, v, w. The air particles having a mean distance r have small displacements with radial component δ. The radial velocity becomes[11]

$$v_r = \frac{\partial \delta}{\partial r} \, . \tag{40}$$

p. 149

The air is considered as an inviscid compressible fluid, with the velocity of sound w_s. The air pressure is $p = p_\infty + q$, where p_∞ is the air pressure at an infinite distance from the cylinder, and q is the small deviation therefrom. All variables are functions of y and z, or r and φ, and of the expression $(x - U_S t)$ in consequence of the motion of the body.

With the higher terms neglected, the Navier-Stokes equations read:

$$\frac{\partial v_r}{\partial t} = -\frac{1}{\rho_L} \frac{\partial q}{\partial r} \, , \quad \frac{\partial u}{\partial t} = -\frac{1}{\rho_L} \frac{\partial q}{\partial x} \, , \quad \frac{\partial v}{\partial t} = -\frac{1}{\rho_L} \frac{\partial q}{\partial y} \, , \quad \frac{\partial w}{\partial t} = -\frac{1}{\rho_L} \frac{\partial q}{\partial z} \, . \tag{41}$$

[11] Editor's note: There is no equation (39) in the original.

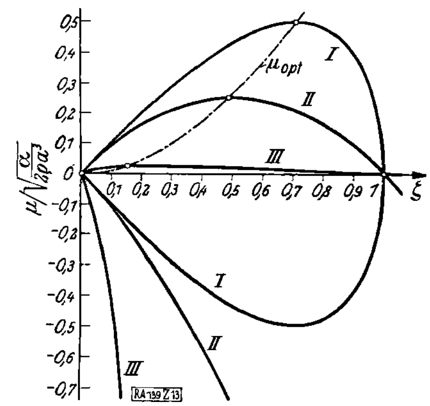

Figure 13: Dependence of $\mu \Big/ \sqrt{\dfrac{a}{2\rho a^3}}$ on ζ for inviscid and viscous fluids. $\dfrac{3}{2}\dfrac{\eta}{\rho a^2} \Big/ \sqrt{\dfrac{\alpha}{2\rho a^3}} = 0$ (curve II, inviscid fluid), = 1 (curve II), and = 20 (curve III).

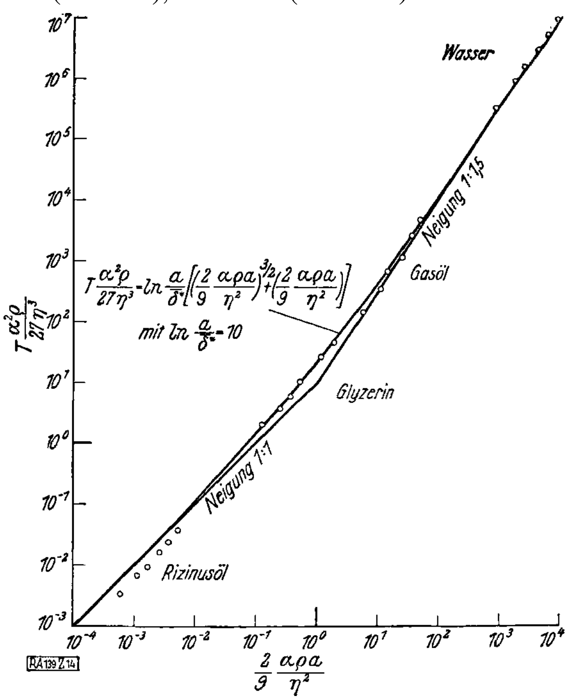

Figure 14: Function $T\dfrac{\alpha^2\rho}{27\eta^3} = \ln\dfrac{a}{\delta^*}\left[\left(\dfrac{2}{9}\dfrac{\alpha\rho a}{\eta^2}\right)^{3/2} + \left(\dfrac{2}{9}\dfrac{\alpha\rho a}{\eta^2}\right)\right]$ and experimental results.

Translator's note: *Rizinusöl* = *Castor oil*. *Neigung* = *Slope*. *Glyzerin* = *Glycerin*. *Gasöl* = *Gas oil*. *Wasser* = *Water*.

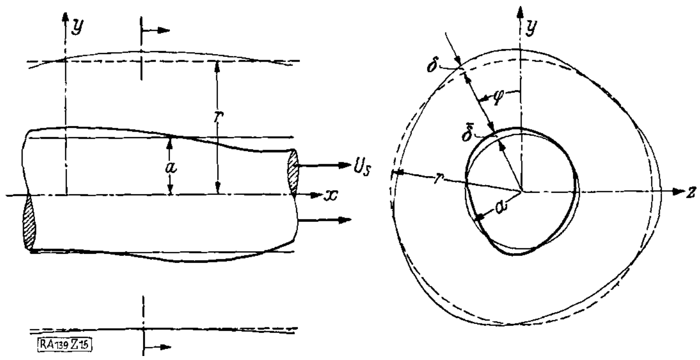

Figure 15: Notations for the derivation of the air effects.

Equations (40) and (41) are satisfied by the substitution

$$\delta = \frac{\partial \Phi}{\partial r}, \quad v_r = \frac{\partial^2 \Phi}{\partial r \partial t}, \quad u = \frac{\partial^2 \Phi}{\partial x \partial t}, \quad v = \frac{\partial^2 \Phi}{\partial y \partial t}, \quad w = \frac{\partial^2 \Phi}{\partial z \partial t}, \quad q = -\rho_{\mathrm{L}} \frac{\partial^2 \Phi}{\partial t^2}. \tag{42}$$

The density ρ_{L} is likewise a magnitude with small variations, so that in

$$\frac{\partial q}{\partial x} = -\rho_{\mathrm{L}} \frac{\partial^3 \Phi}{\partial x \partial t^2} - \frac{\partial \rho_{\mathrm{L}}}{\partial x} \frac{\partial^2 \Phi}{\partial t^2}, \text{ etc.,}$$

the last term, as a product term, may be neglected.

The continuity equation gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{1}{\rho_{\mathrm{L}}} \frac{\partial \rho_{\mathrm{L}}}{\partial t} = 0.$$

The density variation is referred back to the pressure variation in an adiabatic process:

$$p = p_\infty + q = C\rho_{\mathrm{L}}^{\kappa}, \quad C \text{ — constant,}$$
$$\frac{\partial q}{\partial t} = \kappa C \rho_{\mathrm{L}}^{\kappa-1} \frac{\partial \rho_{\mathrm{L}}}{\partial t}.$$

With

$$w_{\mathrm{s}}^2 = \frac{\kappa p}{\rho_{\mathrm{L}}} = \kappa C \rho_{\mathrm{L}}^{\kappa-1} \quad \text{we get} \quad \frac{\partial \rho}{\partial t} = \frac{1}{w_{\mathrm{s}}^2} \frac{\partial q}{\partial t}. \tag{43}$$

If we substitute this value, and q and the velocities from (42) into the continuity equation, we get the differential equation for Φ, wherein we still have to note that

$$\frac{\partial^2}{\partial t^2} f(x - U_S t) = U_S^2 \frac{\partial^2}{\partial x^2} f(x - U_S t),$$

so that

$$\left[\frac{\partial^2}{\partial x^2}\left(1 - \frac{U_S^2}{w_s^2}\right) + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] \Phi = 0,$$

or, in polar coordinates,

$$\left[\frac{\partial^2}{\partial x^2}\left(1 - \frac{U_S^2}{w_s^2}\right) + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right] \Phi = 0. \tag{44}$$

For each solution for Φ we can find the displacement δ and the pressure variation q.

In the following, two special cases will be investigated

1. Φ in polar coordinates independent of φ; rotation-symmetric condition. p. 150

Equation (44) simplifies to

$$\left[\frac{\partial^2}{\partial x^2}\left(1 - \frac{U_S^2}{w_s^2}\right) + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right] \Phi = 0. \tag{45}$$

For a particular Φ solution and correspondingly for δ and q we make the substitution

$$\Phi = \Phi^* \cos \zeta \frac{x - U_S t}{a}. \tag{46}$$

We further set

$$\zeta_1^2 = \zeta^2 \left(1 - \frac{U_S^2}{w_s^2}\right) \tag{47}$$

At velocities small in comparison to the velocity of sound, ζ_1 and ζ will hardly differ. Then equation (45) gives

$$\left[-\frac{\zeta_1^2}{a^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right] \Phi^* = 0.$$

Φ^*, and also q^* in consequence of (42), are Bessel Functions of order zero, of the third kind (Hankel Functions), since $q \to 0$ as $r \to \infty$.

Since $\delta^* = \dfrac{\partial \Phi^*}{\partial r}$, δ^* becomes a Bessel Function of the first order and third kind; again, $\delta \to 0$ as

ζ_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.2	1.4
f_0	0.246	0.367	0.450	0.509	0.558	0.597	0.629	0.657	0.679	0.699	0.734	0.759
f_1	0.098	0.186	0.264	0.332	0.392	0.441	0.487	0.525	0.559	0.589	0.639	0.679

ζ_1	1.6	1.8	2.0	2.5	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0
f_0	0.781	0.799	0.814	0.844	0.864	0.880	0.894	0.912	0.925	0.935	0.943	0.949
f_1	0.712	0.738	0.761	0.804	0.837	0.859	0.874	0.899	0.915	0.925	0.538	0.944

Table 2: $f_0(\zeta_1)$ and $f_1(\zeta_1)$ as functions of ζ_1.

$r \longrightarrow \infty$.

We write $F_{L_0}(z) = iH_0^{(1)}(iz)$ (Hankel Function according to Jahnke-Emde), which is a solution of the differential equation

$$\left(-1 + \frac{\partial^2}{\partial z^2} + \frac{1}{z}\frac{\partial}{\partial z}\right) F_{L_0}(z) = 0,$$

$$F_{L_1}(z) = -F'_{L_0}(z) = -H_1^{(1)}(iz) \quad \text{(Hankel Function according to Jahnke-Emde);}$$

a solution of the differential equation

$$\left(-1 + \frac{\partial^2}{\partial z^2} + \frac{1}{z}\frac{\partial}{\partial z} - \frac{1}{z^2}\right) F_{L_1}(z) = 0,$$

$$-F'_{L_1}(z) = -F''_{L_0}(z) = F_{L_0}(z) + z^{-1} F_{L_1}(z).$$

(Consequence of the differential equation for $F_{L_0}(z)$.)

Then, if the surface deviation $\overline{\delta} = \overline{\delta}^* \cos\zeta \dfrac{x - U_\mathrm{S} t}{a}$, we get

$$\begin{aligned}
\overline{\delta} &= \overline{\delta}^* \cos\zeta \frac{x - U_\mathrm{S} t}{a} \cdot F_{L_1}\left(\zeta_1 \frac{r}{a}\right) \Big/ F_{L_1}(\zeta_1), \\
\Phi &= -\overline{\delta}^* \cos\zeta \frac{x - U_\mathrm{S} t}{a} \cdot a F_{L_0}\left(\zeta_1 \frac{r}{a}\right) \Big/ \zeta_1 F_{L_1}(\zeta_1), \\
q &= -\overline{\delta}^* \frac{\rho_\mathrm{L} U_\mathrm{S}^2 \zeta_1}{a} \cos\zeta \frac{x - U_\mathrm{S} t}{a} \cdot F_{L_0}\left(\zeta_1 \frac{r}{a}\right) \Big/ F_{L_1}(\zeta_1)
\end{aligned}$$

and the surface stress

$$\overline{q} = -\rho_\mathrm{L} \frac{U_\mathrm{S}^2 \zeta_1}{a} \frac{F_{L_0}(\zeta)}{F_{L_1}(\zeta)} \overline{\delta} = -\rho_\mathrm{L} \frac{U_\mathrm{S}^2 \zeta_1}{a} \cdot f_0(\zeta_1) \overline{\delta}. \tag{48}$$

Table 2 gives numerical values of $f_0(\zeta_1)$.

The pressure $\overline{q}$ is positive in the constrictions, negative in the bulges, so that it reinforces the p. 151

drop formation.

2. $\Phi = \Phi_1 \cos\varphi$, Φ_1 is a function of $x - U_\mathrm{S} t$ and r.

Equation (44) simplifies to

$$\left[\frac{\partial^2}{\partial x^2}\left(1 - \frac{U_\mathrm{S}^2}{w_\mathrm{s}^2}\right) + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right]\Phi_1 = 0. \tag{49}$$

For the radial displacements $\delta = \dfrac{\partial \Phi}{\partial r}$ then $\delta = \delta_1 \cos\varphi$ likewise holds. To it corresponds a laterally Check wavy body, in which the unchanged cross-section is pushed up or down by the amount δ_1.

For a particular solution for Φ_1, and correspondingly for δ_1 and q_1, we make the substitution

$$\Phi_1 = \Phi^* \cos\zeta \frac{x - U_\mathrm{S} t}{a} \tag{50}$$

and, as before,

$$\zeta_1^2 = \zeta^2 \left(1 - \frac{U_\mathrm{S}^2}{w_\mathrm{s}^2}\right).$$

Then equation (49) gives

$$\left(-\frac{\zeta_1^2}{a^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\Phi^* = 0.$$

Φ^* is a Bessel Function of the first order and third kind; δ^* is its derivative with respect to r.

For the surface deviation $\overline{\delta}$ we let

$$\overline{\delta} = \overline{\delta}^* \cos\zeta \frac{x - U_\mathrm{S} t}{a} \cos\varphi$$

and with the notations introduced above, get

$$\begin{aligned}
\delta &= \overline{\delta}^* \cos\zeta \frac{x - U_\mathrm{S} t}{a} \cos\varphi \cdot F'_{L_1}\left(\zeta_1 \frac{r}{a}\right) \Big/ F'_{L_1}(\zeta_1),\\
\Phi &= \overline{\delta}^* \cos\zeta \frac{x - U_\mathrm{S} t}{a} \cos\varphi \cdot a F_{L_1}\left(\zeta_1 \frac{r}{a}\right) \Big/ \zeta_1 F'_{L_1}(\zeta_1),\\
q &= \overline{\delta}^* \rho_\mathrm{L} \frac{U_\mathrm{S}^2}{a} \zeta_1 \cos\zeta \frac{x - U_\mathrm{S} t}{a} \cos\varphi \cdot F_{L_1}\left(\zeta_1 \frac{r}{a}\right) \Big/ F'_{L_1}(\zeta_1),
\end{aligned}$$

and the surface stress Check

$$\overline{q} = -\rho_{\mathrm{L}} \frac{U_{\mathrm{S}}^2 \zeta_1}{a} \frac{F_{L_1}(\zeta_1)}{-F'_{L_1}(\zeta_1)} \overline{\delta} = -\rho_{\mathrm{L}} \frac{U_{\mathrm{S}}^2 \zeta_1}{a} f_1(\zeta_1) \overline{\delta}. \tag{51}$$

Table 2 likewise contains numerical values of

$$f_1(\zeta_1) = \frac{F_{L_1}(\zeta_1)}{-F'_{L_1}(\zeta_1)} = 1 \Bigg/ \left(\frac{F_{L_0}(\zeta_1)}{F_{L_1}(\zeta_1)} + \frac{1}{\zeta_1} \right) = 1 \Bigg/ \left(f_0(\zeta_1) + \frac{1}{\zeta_1} \right).$$

Due to the surface stress, the wave formation of the jet is excited.

In fluid jets which are emitted from a nozzle, the waviness increases with distance from the nozzle. It remains, however, constant at each point in the air space, so that the solutions of case 1 and 2 can be taken also for the nozzle jets.

7 Drop formation with air effects

Stress due to the effect of the air according to equation (48) must be added to the stress due to surface tension. In place of equation (21) we get for μ the equation

$$\mu^2 + \mu \frac{3\eta}{\rho a^2} \zeta^2 = \frac{\alpha}{2\rho a^3}(1 - \zeta^2)\zeta^2 + \frac{\rho_{\mathrm{L}} U_{\mathrm{S}}^2}{2\rho a^2} \zeta^3 f_0(\zeta). \tag{52}$$

Here we have put $\zeta_1 = \zeta$, on the assumption that U_{S} is small in comparison with the velocity of sound w_{s}.

These μ values will also be taken for the jet which issues from a nozzle. The calculation is carried further for water as an inviscid fluid with:

Jet radius $a = 0.025$ cm, $\alpha = 0.072$ g cm^{-1}, $\rho = 0.00102$ g sec^2 cm^{-4}, $\rho_{\mathrm{L}}/\rho = 0.00129$.

We set $U_{\mathrm{S}} = 0, = 500, = 1000, = 1500$ and 2000 cm sec^{-1} and calculate the μ values for various ζ. p. 152
Figure 16 shows the outcome. The μ values become greater with increasing U_{S} and also appear for greater ζ. While without air effect $\zeta_{\max} = 1$, corresponding to $l_{\min} = 2\pi a = 6.3a$, for 1500 cm sec^{-1} we get $\zeta_{\max} \approx 1.4$, corresponding to $l_{\min} \approx 1.4\pi a = 4.4a$. Likewise, the most favorable values for breakup, ζ_{opt} and μ_{opt} increases: while for smaller velocities $\zeta_{\mathrm{opt}} \approx 0.7$, $\mu_{\mathrm{opt}} \approx 740$ sec^{-1}, we get for 1500 cm sec^{-1} $\zeta_{\mathrm{opt}} \approx 1.1$ and $\mu_{\mathrm{opt}} \approx 1300$ sec^{-1}. The drop length must become shorter, which is also to be observed.

If we assume that at the nozzle only initial disturbances, corresponding to the current value ζ_{opt} and of equal initial strength, occur, then the breakup times and lengths become, respectively

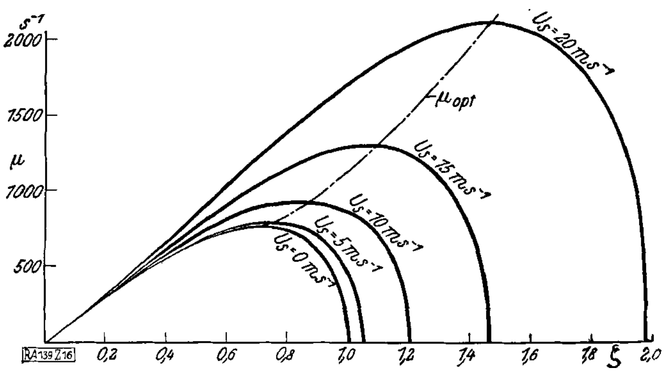

Figure 16: Dependence of μ on ζ for a water jet, $a = 0.025$ cm velocities $U_\mathrm{S} = 0, = 500, = 1000, =$ 1500 and = 2000 cm/sec.

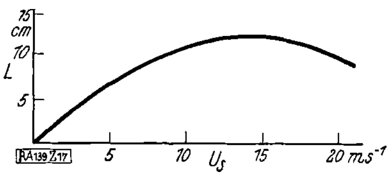

Figure 17: Theoretical breakup length L of a water jet ($a = 0.025$ cm) with different jet velocities.

$T = \dfrac{1}{\mu_\mathrm{opt}} \ln \dfrac{a}{\overline{\delta}^*}$ and $L = TU_\mathrm{S} = \dfrac{U_\mathrm{S}}{\mu_\mathrm{opt}} \ln \dfrac{a}{\overline{\delta}^*}$.

With $\ln \dfrac{a}{\overline{\delta}^*} = 12$, as determined by the experiments for low velocities, we get, for $a = 0.025$ cm, L as shown in Figure 17.

The maximum value of the breakup length occurs from $U_\mathrm{S} = 1500$ cm sec^{-1}; the observations show a lower maximum, which occurs already for 600 cm sec^{-1}. This is to be referred back to the fact that at the breakup the cosine disturbances are concerned with all ζ values and these have different initial strengths, dependent also on U_S. On the contrary, in a similar investigation of the drop formation with very viscous fluids, it was found that the calculated breakup length was less than the observed. This was explained by the air being carried along with the jet.

In the experiments surface wave formation was shown with velocities under 500 cm sec^{-1} with $\zeta > 3$; this can not be explained by the effect of the air, but is to be attributed to periodic helical vortices which have formed in the nozzle.

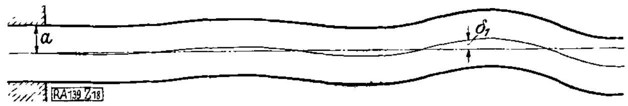

Figure 18: Wave formation for a jet issuing from a nozzle.

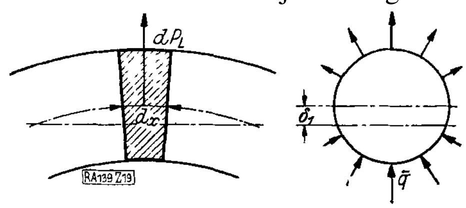

Figure 19: Force $\mathrm{d}P_L$ resulting from the effect of the air.

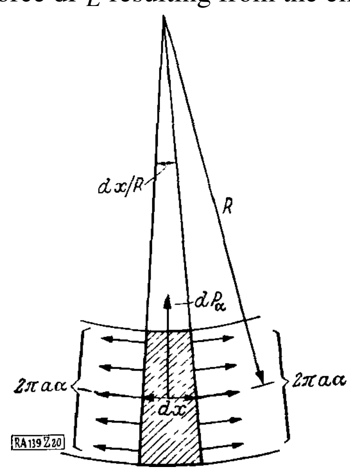

Figure 20: Force $\mathrm{d}P_L$ resulting from surface tension.

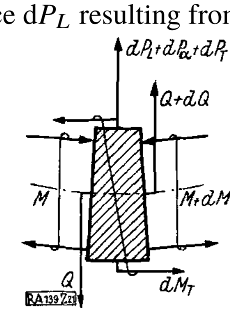

Figure 21: Bending moment and transverse forces in jet element $\mathrm{d}x$. Check

8 Wave formation by air effects

p. 153

In wave formation the cross-section of the jet does not vary; the middle line (line of centers of the cross-sections) is subject to small deviations δ_1 from the straight axis. The corresponding disturbances at the nozzle will be expanded into a Fourier series. Each individual disturbance gives a wavy line. If we join the coordinate system to the jet, we get (Figure 18) for a single disturbance

$$\delta_1 = \delta^* e^{\mu(t+x/U_S)} \cdot \cos \zeta \frac{x}{a}.$$

For the determination of μ a longer jet with a wave form $\delta_1 = \delta^* e^{\mu t} \cos \zeta \frac{x}{a}$ varying with the time is investigated for drop formation; the wave moves through the air with velocity U_S. The permissibility of this simplification has been shown in connection with drop formation.

The forces acting on the jet arise from the effect of the air and from surface tension. The effect of the air is given by $\overline{q}$ (equation (51), where we have assumed $U_S < w_s$, and have set $\zeta_1 = \zeta$. On the piece of the jet dx we get the forces (Figure 19).

$$\mathrm{d}P_L = \int_0^{2\pi} (-\overline{q}\,\mathrm{d}x \cos^2 \varphi a) d\varphi = \pi \rho_L U_S^2 \zeta f_1(\zeta) \delta_1 \,\mathrm{d}x.$$

The surface tension produces on the section dx, by equation (1) or immediately from Figure 20, the force Check

$$\mathrm{d}P_\alpha = \frac{2\pi a \alpha}{R}\,\mathrm{d}x = 2\pi a \alpha \delta_1''\,\mathrm{d}x = -\frac{2\pi\alpha}{a}\zeta^2 \delta_1 \,\mathrm{d}x.$$

The jet can be considered as a beam, subject to shear and bending. The bending moment at point x is $M = M^* \cos \zeta \frac{x}{a}$, with transverse force $Q = Q^* \sin \zeta \frac{x}{a}$; signs are shown in Figure 21.

From this we obtain $\delta_1 = \delta_M + \delta_Q$ for the displacement of the centerline, if the known formulas of the elastic problem are correspondingly changed:

$$\frac{M}{3J\eta} = \frac{\partial^3 \delta_M}{\partial x^2 \partial t} = -\frac{\zeta^2}{a^2}\mu\delta_M$$
$$\frac{\partial Q}{\partial x}\frac{1.2}{F\eta} = \frac{\partial^3 \delta_M}{\partial x^2 \partial t} = -\frac{\zeta^2}{a^2}\mu\delta_Q.$$

The element dx has the acceleration $\frac{\partial^2 \delta_1}{\partial t^2}$ and the angular acceleration $\frac{\partial^3 \delta_M}{\partial t^2 \partial x}$.

Using D'Alembert's Principle we introduce the inertial forces (Figure 18):

$$\begin{aligned} \mathrm{d}P_T &= -\frac{\partial^2 \delta_1}{\partial t^2}\,\rho F\,\mathrm{d}x = -\mu^2 \rho F \delta_1\,\mathrm{d}x \\ \mathrm{d}M_T &= -\frac{\partial^3 \delta_M}{\partial t^2 \partial x}\,\rho J\,\mathrm{d}x = -\mu^2 \rho J\,\frac{\partial \delta_M}{\partial x}\,\mathrm{d}x. \end{aligned}$$

The conditions of equilibrium give:

$$\begin{aligned} \mathrm{d}P_\alpha + \mathrm{d}P_L + \mathrm{d}P_T + \mathrm{d}Q &= 0, \\ \mathrm{d}M_T + \mathrm{d}M + Q\,\mathrm{d}x &= 0. \end{aligned}$$

(The condition is being investigated statically for a given instant, so that we write $\mathrm{d}M$ instead of $\frac{\partial M}{\partial x}\,\mathrm{d}x$, etc.)

If all magnitudes except δ_1 are eliminated from this system of equations, then we obtain with $J = \pi a^4/4$ and $F = \pi a^2$:[12]

$$\mu^2 + \mu \frac{\eta \zeta^2}{1.2\rho a^2} \frac{3\frac{\eta\zeta^2}{\rho a^2} + \mu}{\frac{10}{3}\frac{\eta}{\rho a^2} + 3\frac{\eta\zeta^2}{\rho a^2} + \mu} = \frac{\rho_\mathrm{L}}{\rho}\frac{U_\mathrm{S}^2}{a^2}\zeta f_1(\zeta) - \frac{2\alpha}{a^3\rho}\zeta^2. \tag{52*}$$

On the right-hand side are the terms which produce or prevent the breakup. With inviscid fluids only μ^2 remains on the left; this is the maximum μ for a given right-hand wide. Because of viscosity μ will be smaller. Equation (52*) may be reduced to an equation of the third degree and the calculation based on it is quite convenient. We calculate a first approximation to μ by neglecting the second term on the left. Then we put this μ into the second term, find a second approximation, and so on. p. 154

For an example, let a glycerin jet be taken with $a = 0.025$ cm, $\rho = 0.00125$ g sec^2 cm^{-4}, $\alpha = 0.066$ g cm^{-1}, $\eta = 9.3 \cdot 10^{-4}$ g sec m^{-2}, $\rho_\mathrm{L}/\rho = 0.00106$. For each ζ there must be a minimum velocity $U_{\mathrm{S}_1\,\mathrm{min}}(\zeta)$. Indeed, the right-hand side must be greater than or equal to zero:

$$U_{\mathrm{S}_1\,\mathrm{min}}(\zeta) = \sqrt{\frac{2\alpha\rho\zeta}{a\rho\rho_\mathrm{L} f_1(\zeta)}} = 2000\sqrt{\frac{\zeta}{f_1(\zeta)}}\ \mathrm{cm\ sec}^{-1}.$$

We get

[12] Editor's note: Equation (52) is duplicated in the original, so this is numbered 52*.

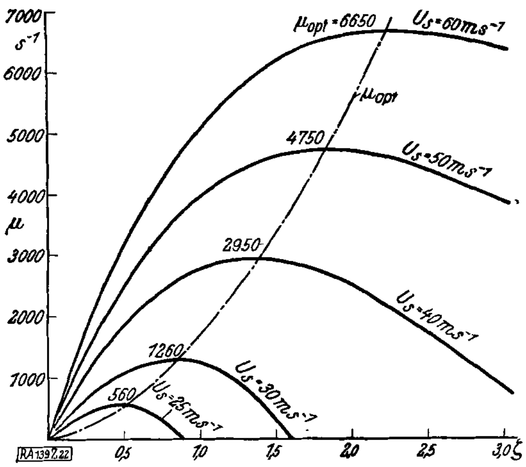

Figure 22: Dependence of μ on ζ in wave formation for a glycerin jet with $a = 0.025$ cm and for jet velocities $U_S = 25, = 30, = 35, = 40$ and $= 50$ m/sec.

$\zeta =$	0	0.4	0.8	1.2	1.6	2.0	2.5	3.0	
$U_{S_{1\,\min}}(\zeta) =$	20	20.4	24.7	27.4	30.0	32.4	35.2	37.9	m sec^{-1}.

For $\zeta = 0$ we obtain the minimum velocity for which wave formation occurs, $U_{S_{1\,\min}}(0)$.

With increasing velocity, we get first long, then shorter waves. With a jet radius $a = 0.025$ cm we have $U_{S_{1\,\min}}(0) \approx 20$ m sec^{-1} for water and glycerin and ≈ 15 m sec^{-1} for gas oil and castor oil, so that with the latter fluids wave formation is to be observed at lower velocities.

The μ values corresponding to the velocities $U_S = 25$ m sec^{-1}, 30 m sec^{-1}, 40 m sec^{-1} and 50 m sec^{-1} are calculated by equation (52*) and represented in Figure 22.

For a given velocity, we get the maximum value μ_{opt} for ζ_{opt}. If the initial disturbances are present in equal magnitude for all ζ, μ_{opt} will prevail most strongly; ζ_{opt} increases with increasing velocity, that is, the wavelength decreases, a fact which is also observed. If we assume that the initial disturbances at the nozzle correspond only to a particular value of ζ_{opt}, that δ^* always has the same value at the nozzle, and that δ^* increases until breakup according to the exponential law up to h, a certain multiple of a, then the breakup time becomes $T = \dfrac{1}{\mu_{\text{opt}}}\left(\ln \dfrac{h}{\delta^*_{\text{nozzle}}}\right)$, and the breakup length becomes $L = U_S\, T$. Check Check

For example, we get

$U_S =$	20	25	30	40	50	60	m sec^{-1}
$L/\ln\left(\dfrac{h}{\delta^*_{\text{nozzle}}}\right) =$	∞	$\frac{1}{22}$	$\frac{1}{430}$	$\frac{1}{720}$	$\frac{1}{1060}$	$\frac{1}{1060}$	m.

With increasing U_S, the wavelength first decreases rapidly, then more slowly, as it is also observed. Check
Here also there appear discrepancies between calculation and experiment, since the adjacent air is carried along with the jet.

9 Concluding remarks

The breakup phenomena of a jet which can be observed at moderate velocities — drop and wave formation — have been investigated theoretically.

Rotation-symmetric disturbances lead to formation of drops. The breakup time for low velocities is for a given fluid and jet diameter are invariant, while the breakup length increases with the velocity. Check
The breakup time and the wavelength of drop formation depend upon the physical magnitudes of the jet; theory and experiment show good agreement. Under the influence of air forces the breakup time becomes theoretically smaller, so that with increasing velocity the breakup length also decreases, although the experiments show a sharper decrease.

Wave formation is explained by the influence of aerodynamic forces and theoretical laws are derived which agree in character with the experimental results.

Nomenclature[13]

a =	Radius of cylinder or mean radius, L
α =	Surface tension, F/L
$\bar{\delta}$ =	Deviation from cylindrical form of radius a to jet surface, L
q_α =	Local deviation from pressure previously considered uniform, F/L^2
$1/R_1$ and $1/R_2$	are curvatures in two mutually perpendicular planes, 1/L
x =	Component in direction of motion, L
φ =	Angular deviation from vertical in plane perpendicular to direction of motion, °
r =	Actual radius of jet at any point, L
l =	Wavelength, L
ζ =	Radio of the circumference of the jet to the wavelength = $\dfrac{2\pi a}{l}$
$\bar{\delta}^*$	Maximum deviation from the mean radius, L
η =	Viscosity, M/LT
t =	Time, T

[13] Editor's note: This section is not in the original and is copied from the 1948 translation.

u = Linear velocity in x direction, L/T

v = Radial velocity in plane perpendicular to x, L/T

ρ = Density, M/L^3

$\overline{\delta}$ = Deviation from the average radius at any point in the flow, L

δ_1 and δ_2 represent two solutions for a differential equation, L

τ = Shear stress in fluid, F/L^2

τ = Shear stress at the boundary, F/L^2

μ = Forcing or damping unit, 1/T

δ^* Maximum deviation from mean radius r, L

$F_1(m)$ = Bessel Function = $-iJ_1(im)$

$F_0(m)$ = Bessel Function = $-iJ_0(im)$

p_x = Pressure in x direction, F/L^2

ν = Poisson's no.

G = Shear modulus, F/L^2

$\sigma_x, \sigma_y, \sigma_z$ = Normal Stresses in x, y, z direction, F/L^2

τ_{xy} = Shear stress in z plane, F/L^2

V = Specific volume, L^3/M

ζ_{opt} = Value of ζ corresponding to μ_{opt}

μ_{opt} = Maximum value of μ, 1/T

l_{opt} = Wavelength corresponding to μ_{opt}, L

U_{S} = Velocity of jet as it leaves nozzle, L/T

U_1 = Velocity of jet at any point, L/T

ε = $\dfrac{1}{U}\sqrt{\dfrac{\alpha}{\rho a}}$

v_{S} = $\zeta U_1/a$

T = L/U_{S}, breakup time, T

L = Breakup length, L

$\overline{\delta}_1$ = Disturbances at l_{opt}, L

$\overline{\delta}_{10}$ = Disturbances at $10l_{\text{opt}}$, L

S_1, S_{10} = Magnitude of singularity

v_r = Radial velocity of air, L/T

w_{s} = Velocity of sound in air, L/T

p_∞ = Pressure of air at infinity, L/T

q = Small pressure deviation from p at jet surface, F/L^2

ρ_{L} = Density of air, M/L^3

ϕ = A field value having dimension L^2; similar to the velocity potential, $\phi = f(x, r, \varphi)$

ζ_1^2 = $\zeta^2(1 - U_{\mathrm{S}}^2/w_{\mathrm{s}}^2)$

$F_{L_0}(z)$ = $iH_0^{(1)}(iz)$ Hankel Function of the first order and third kind

δ_1 = The deviation of an unchanged cross-section from the x axis, L

$f_0(\zeta_1)$ = $F_{L_0}(\zeta_1)/F_{L_1}(\zeta_1) = iH_0^{(0)}(i\zeta_1)/-H_1^{(1)}(i\zeta_1)$

$f_1(\zeta_1)$ = $\dfrac{1}{f_0(\zeta_1) + 1/\zeta_1}$

P_L = Summation of air forces, F

P_α = Summation of tension forces, F

P_T = Inertial forces, F

M = Bending Moment, FL

Q = Transverse force, F

Φ = $\phi\, f(x, r)$ only, L^2[14]

Φ^* = Maximum value of $\phi\, f(x, r)$, L^2

q^* = Maximum pressure deviation from p_∞ at surface of jet, F/L^2

[14]Editor's note: Written as printed. Unclear what the notation $\phi f(x, r)$ means in this context. Perhaps $\phi f(x, r)$ means ϕ is only a function of x and r?